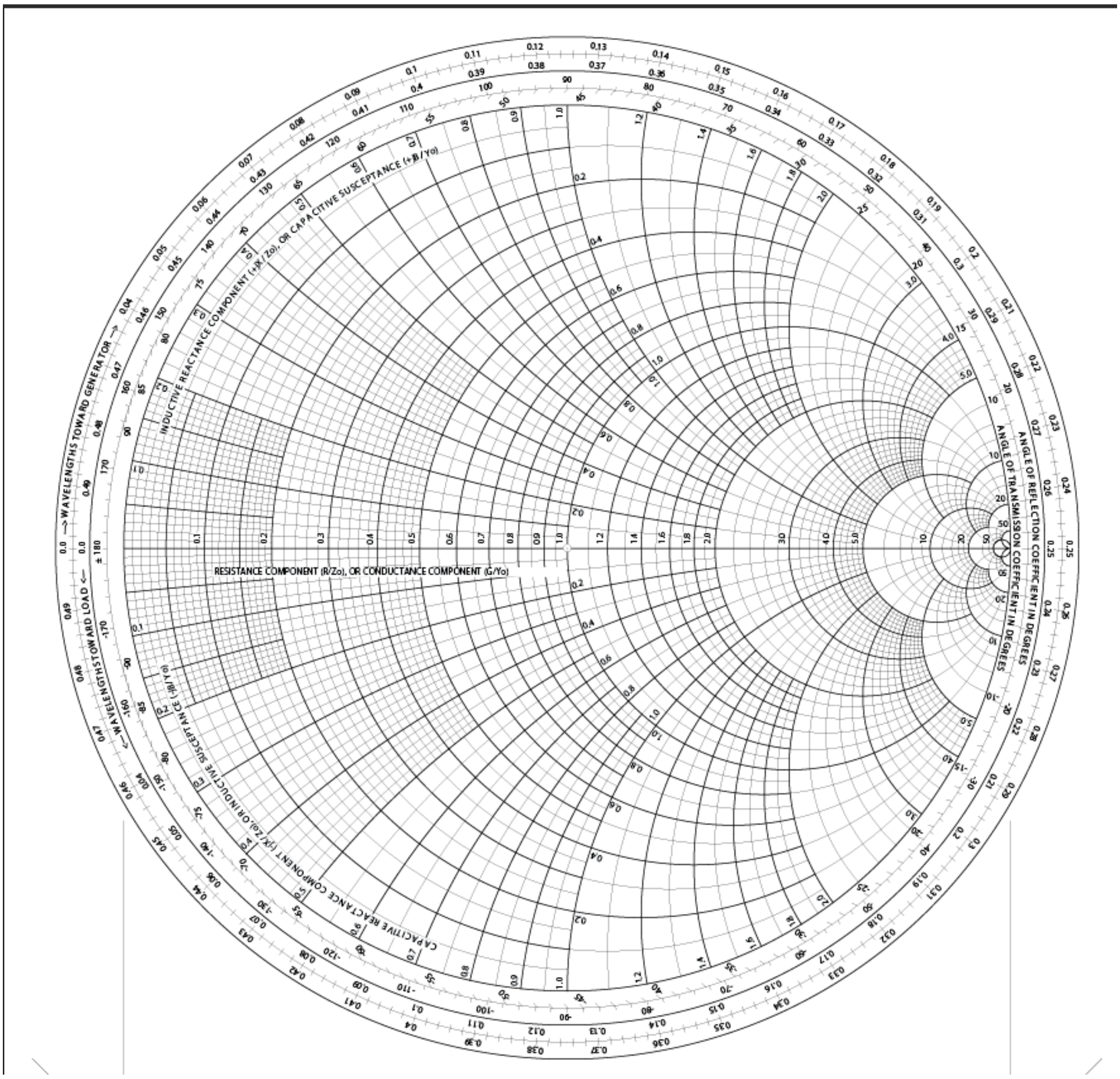
# TRANSMISSION LINES
# SOLVED EXAMPLES






P.Viljakainen
January 2021

## Table of contents





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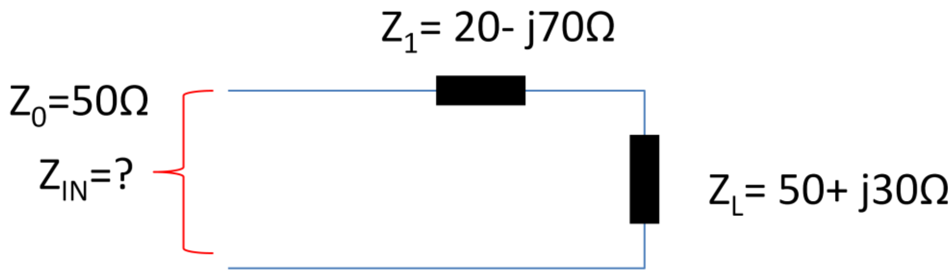
## 1. Impedances in Series

Example: Find $Z_{IN}$

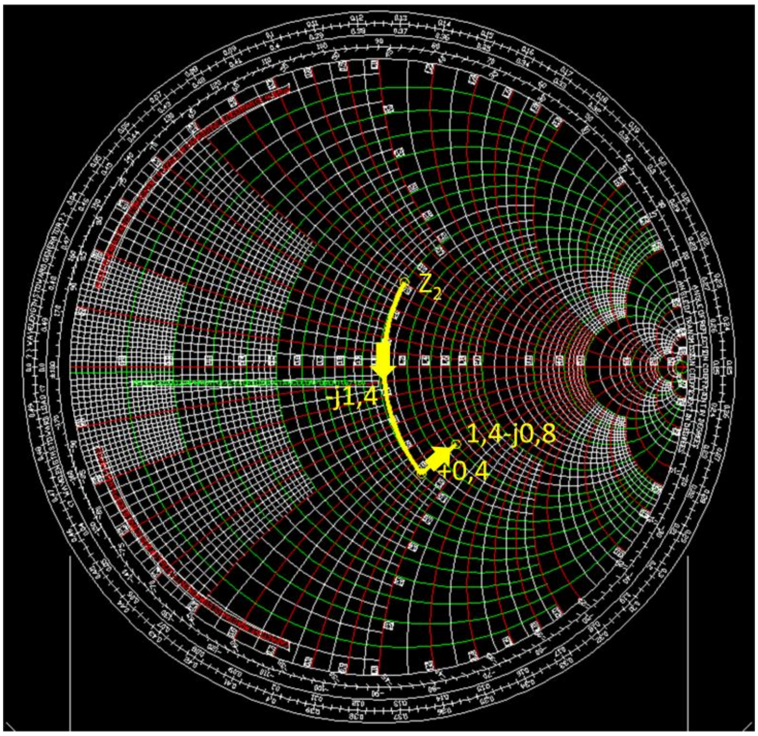



### 1.1 By calculation

$Z_{IN}$ = $Z_1$ + $Z_L$ = (20-j70)Ω +(50 +j30)Ω = **(70-j40)Ω = 80.6 <-29,7° Ω.**

### 1.2 Using Smith Chart

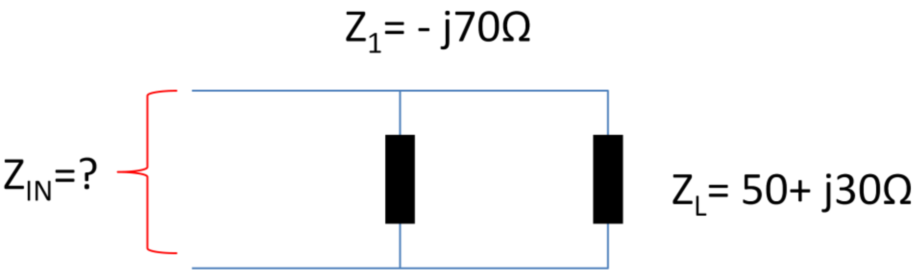
- Normalize ; $Z_L$'= $Z_L$/50, $Z_1$'= $Z_1$/50,
- $Z_L$'=1+ j0,6Ω ->find point on the Smith chart.
- $Z_1$'=0,4- j1,4Ω (resistance and capacitance in series)
    - on a constant resistance circle, from ZL', rotate counterclockwise 1,4 units to add series C
    - on a constant reactance arc, move 0,4 units to add a series R
- Convert back to system impedance
- $Z_{IN}$=( 50 x 1,4)Ω + (50 x-j0,8)Ω = 70-j40Ω




**Kuva 1: Impedances in Series**

## 2. Impedances in Parallel

Example: Find $Z_{IN}$




### 2.1 By calculation

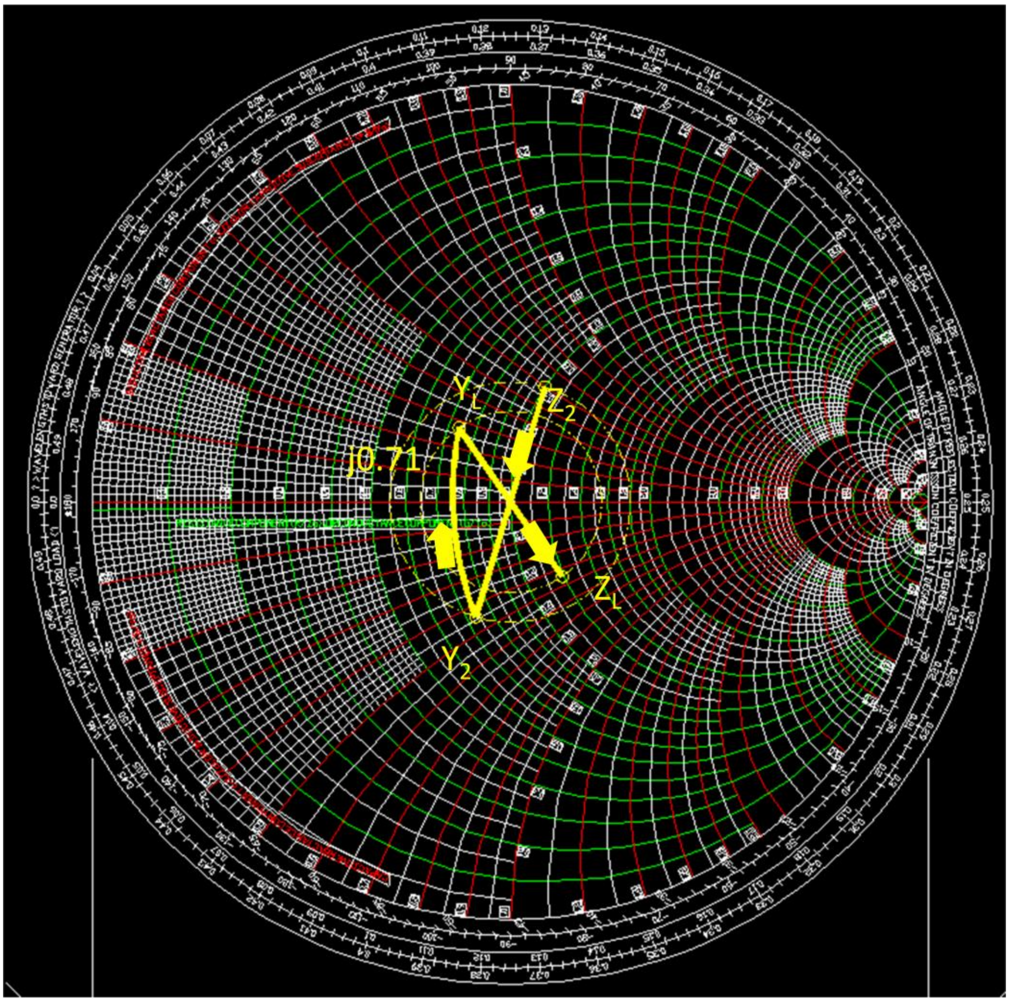
$$Z_{IN} = \frac{1}{\frac{1}{Z_1}+\frac{1}{Z_L}} = \frac{1}{\frac{1}{-j70}+\frac{1}{50+j30}} = \frac{1}{\frac{1}{70\angle-90°}+\frac{1}{58{,}3\angle30{,}96°}}$$

$$= \frac{1}{0{,}0143\angle90°+0{,}0172\angle-30{,}96°} = \frac{1}{0+j0{,}0143+0{,}0147-j0{,}0088}$$

$$= \frac{1}{0{,}0147+j0{,}0055} = \frac{1}{0{,}0157\angle20{,}5°} = 63{,}7\angle-20{,}5°\Omega = \mathbf{59{,}7-j22{,}3\Omega}$$

### 2.2 Using Smith Chart

- Normalize ; $Z_L' = Z_L/50$, $Z_1' = Z_1/50$,
- $Z_L'$=1+ j0,6Ω ->find point on the Smith chart.
- Find $1/Z_L' = Y_L'$ on the smith chart by "mirroring"
- $Z_1'$=- j1,4Ω - > Calculate $Y_1$ = 1/1,4∠90 = 0+j0,71S
    - On a constant conductance circle, from $Y_1$ rotate clockwise 0,71 units to arrive at $Y_L$
- Convert to $Z_L$ =1,19 –j0,45 by "mirroring"
- Convert back to system impedance
- **$Z_{IN}$**=( 50 x 1,19)Ω + (50 x-j0,45)Ω ≈ **59,7-j22,3Ω**

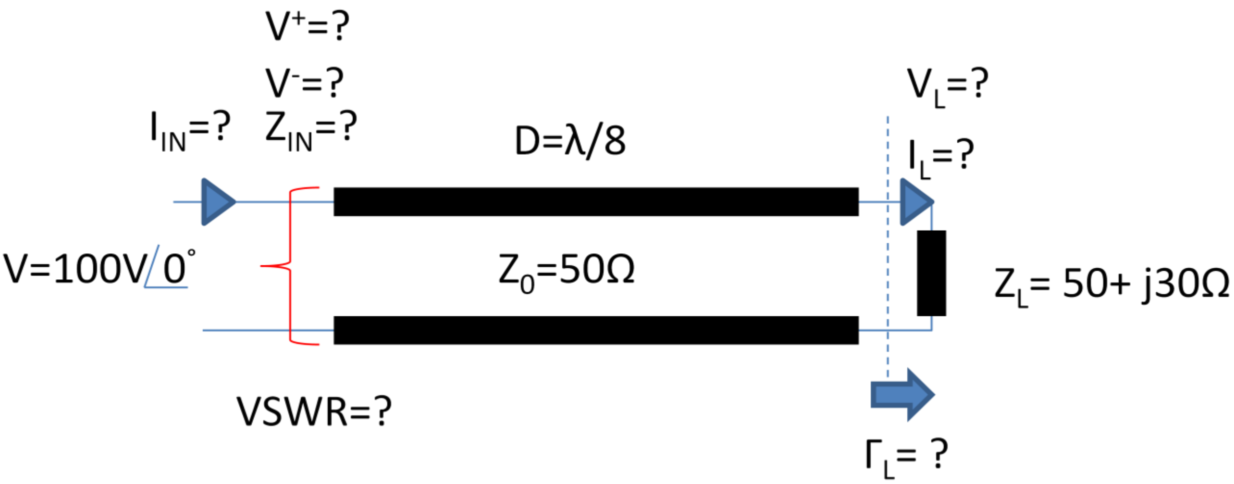



**Kuva 2:Impedances in Parallel**

## 3. Lossless transmission line, terminated with an arbitrary impedance

### 3.1 Example:

Find $Z_{IN}$, $I_{IN}$, $V_L$ , $I_L$ , VSWR as a ratio and in dB, return loss in dB, complex voltage/current reflection coefficient and power reflection coefficient at load of the following circuit. Transmission line which length is λ/8 is assumed to be lossless. System impedance is 50Ω.




## 3.2 By calculation

Input impedance for a lossless transmission line terminated with a complex impedance is

$$\boxed{Z_{IN} = Z_0 \frac{Z_L + jZ_0 tan\beta D}{Z_0 + jZ_L tan\beta D}}$$

**Kaava 1**

where

$Z_0$ = system impedance;
$Z_L$ = load impedance;
$\beta$ = wave number = 2π/λ;
$\lambda$= wavelength;
$D$ = length of transmission line;

Wavelength is

$$\lambda = \frac{v_p}{f}$$

Where

$v_p$ = Speed of propagation in the transmission line;
$f$ = Frequency;

$$\boldsymbol{Z_{IN}} = 50 \frac{50 + j30 + j(50\,tan\frac{\pi}{4})}{50 + j(50 + j30)(tan\frac{\pi}{4})} = 50 \frac{50 + j80}{50 + j50 - 30} = \frac{2500 + j4000}{20 + j50} = \frac{4716\angle 58{,}0°}{53{,}85\angle 68{,}2°}$$

$$= 86{,}2 - j15{,}5\ \Omega = \boldsymbol{87{,}6\angle - 10{,}2°\ \Omega}$$

and

$$\boldsymbol{I_{IN}} = \frac{V_{IN}}{Z_{IN}} = \frac{100\angle 0°}{87{,}6\angle - 10{,}2°} = \boldsymbol{1{,}14\angle 10{,}2°\ A}$$

Wave functions for voltage and current as function of distance z (z=0 at generator side of the line) in lossless transmission line are

$$V(z) = V^+ e^{(-j\beta z)} + V^- e^{(j\beta z)}$$

$$I(z) = \frac{V^+}{Z_0} + e^{(-j\beta z)} - \frac{V^-}{Z_0} e^{(j\beta z)}$$

Where

$V^+$ = forward propagating voltage wave
$V^-$ =backward propagating voltage wave

at z=0 voltage and current must be

$$V(0) = V^+ + V^- = 100V\angle 0°$$
$$I(0) = I_{IN} = \frac{V^+}{Z_0} - \frac{V^-}{Z_0} = 1{,}14\angle 10{,}2° A$$

Solve V(0) and I(0) from above by substitution:

$$I(0) = \frac{100\angle 0° - V^-}{50} - \frac{V^-}{50} = 1{,}14\angle 10{,}2° =>$$

$$\boldsymbol{V^-} = \frac{100\angle 0° - 57\angle 10{,}2°}{2} = \mathbf{22{,}52\angle - 12{,}95° V} = 21{,}95 - j5{,}05\ V$$

$$\boldsymbol{V^+} = 100\angle 0° - 22{,}52\angle - 12{,}95° = 78{,}05 + j5{,}05 = \mathbf{78{,}21\angle + 3{,}7° V}$$

And check

$$V^+ + V^- = 78{,}05 + j5{,}05 + 21{,}95 - j5{,}05 = 100\angle 0° V; OK$$

Solve $V_L$ and $I_L$ at distance D=βD=π/4

$$V(D) = V^+ e^{(-j\pi/4)} + V^- e^{(j\pi/4)}$$

$$I(D) = \frac{V^+}{Z_0} e^{(-j\pi/4)} - \frac{V^-}{Z_0} e^{(j\pi/4)}$$

using Euler's formula

$$e^{j\varphi} = cos\varphi + jsin\varphi$$

$$V(D) = V^+(cos\left(\frac{-\pi}{4}\right) + jsin\left(\frac{-\pi}{4}\right)) + V^-(cos\left(\frac{\pi}{4}\right) + jsin\left(\frac{\pi}{4}\right))$$
$$I(D) = \frac{V^+}{Z_0}\left(cos\left(\frac{-\pi}{4}\right) + jsin\left(\frac{-\pi}{4}\right)\right) - \frac{V^-}{Z_0}\left(cos\left(\frac{\pi}{4}\right) + jsin\left(\frac{\pi}{4}\right)\right)$$

$$\boldsymbol{V(D)} = (78{,}05 + j5{,}05)(0{,}707 - j0{,}707) + (21{,}95 - j5{,}05)(0{,}707 + j0{,}707)$$

$$= 77{,}84 - j39{,}66 = \mathbf{87{,}36\angle - 27° V}$$

$$\boldsymbol{I(D)} = \frac{78,05 + j5,05}{50}\left(cos\left(\frac{-\pi}{4}\right) + jsin\left(\frac{-\pi}{4}\right)\right) - \left(\frac{21,95 - j5,05}{50}\left(cos\left(\frac{\pi}{4}\right) + jsin\left(\frac{\pi}{4}\right)\right)\right)$$
$$= 1,556 + j0,1)(0,7,7 - j0,707) - ((0,44 - j0,10)(0,707 + j0,707)$$
$$= 0,79 - j1,27 = \mathbf{1,49\angle - 58,1°}\ \boldsymbol{A}$$

Check that impedance at load is correct;

$$Z_L = \frac{V(D)}{I(D)} = \frac{87,36\angle - 27°V}{1,49\angle - 58,1°A} = 58,63\angle 31,9° \approx 50 + j30\Omega$$

Voltage or current Reflection coefficient is (at load)

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$\boldsymbol{\Gamma_L} = \frac{50 + j30 - 50}{50 + j30 + 50} = \frac{30\angle 90°}{104.4\angle 16.7} = \mathbf{0,287\angle 73,3°}$$

and at generator

$$\Gamma_G = \frac{Z_{IN} - Z_0}{Z_{IN} + Z_0}$$

$$\boldsymbol{\Gamma_G} = \frac{86,2 - j15,5\ - 50}{86,2 - j15,5\ + 50} = \frac{39,4\angle - 23.1°}{137,0\angle - 6,49°} = \mathbf{0,287\angle - 16,6°}$$

Return Loss in dB is

$$R = 20\ \log|\Gamma|$$

$$\boldsymbol{R} = 20\ \log\ 0,287 = \mathbf{-10,7}\ \boldsymbol{dB}$$

Power Reflection coefficient is

$$|s|^2 = \left|\frac{Z_L - Z_0}{Z_L + Z_0}\right|^2$$

$$|\boldsymbol{s}|^{\mathbf{2}} = \left|\frac{50 + j30 - 50}{50 + j30 + 50}\right|^2 = 0,297\ ^2 = \mathbf{0,082}$$

Voltage Standing Wave Ratio is

$$VSWR = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|}$$
$$\frac{VSWR}{dB} = 20 \log\left(\frac{1 + |\Gamma_L|}{1 - |\Gamma_L|}\right)$$

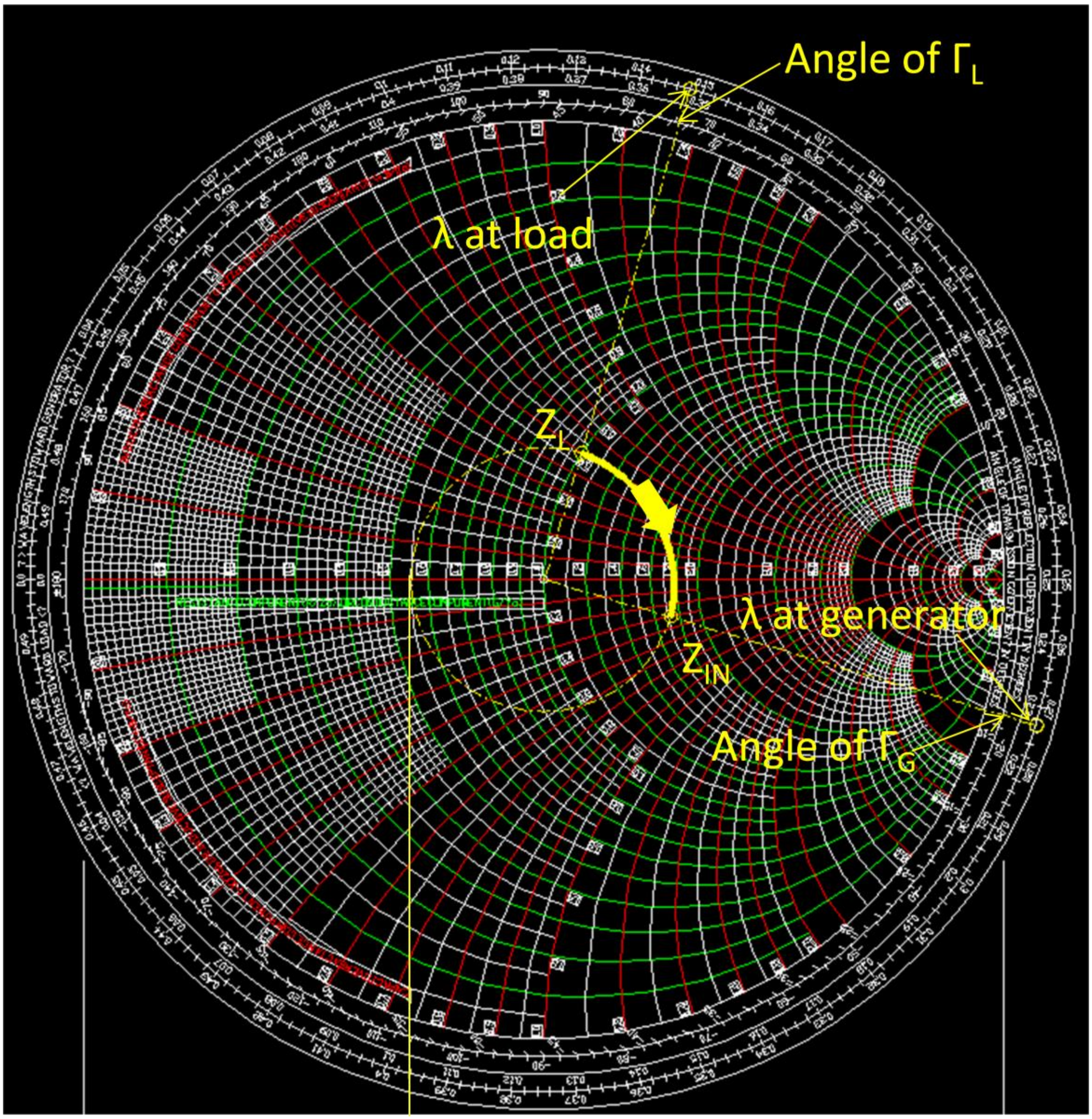
$$\boldsymbol{VSWR} = \frac{1 + 0,287}{1 - 0,287} = \mathbf{1,82}$$
$$\boldsymbol{VSWR} = 20\ \log 1,82 = \mathbf{5,2}\ \boldsymbol{dB}$$

## 3.3 Using Smith Chart

Many of the above parameters can be read directly from the Smith Chart without calculation and others calculated by using them.

1. Normalize ; $Z_L$'= $Z_L$/50, $Z_L$'=1+ j0,6Ω ->find point on the Smith chart
2. Draw a circle to that point with its center in the center of the chart
3. From center point extend a line through $Z_L$' to the scale "wavelengths towards generator"-> read "λ at load# =0,148
4. Move on the scale towards generator (clockwise) and add transmission line's λ/8=0,125 units and arrive to "λ at generator"=0,273
5. Draw a line from point λ at generator to the center of the Smith Chart, read $Z_{IN}$'=1,83-j0,3
6. Convert $Z_{IN}$' back to system impedance level 50Ω

$Z_{IN}$=50x1,74- 50x j0,3≈87-j15Ω




**Kuva 3:Transmission line**



P.Viljakainen
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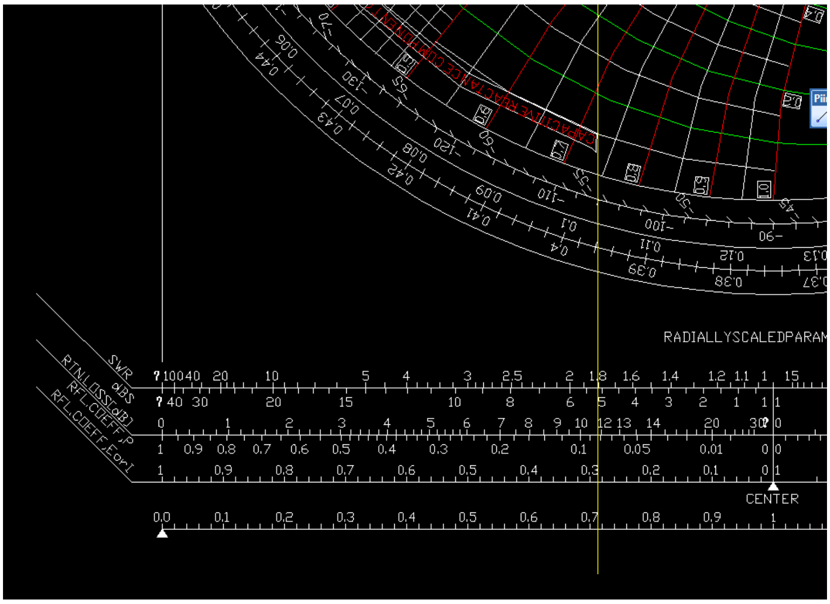
- From the circle drawn above, draw from the crossing of the circle and the horizontal axis a straight line down to “radially scaled parameters”



Kuva 4:Radially scaled parameters, SWR, Return Loss, Reflection Coefficient

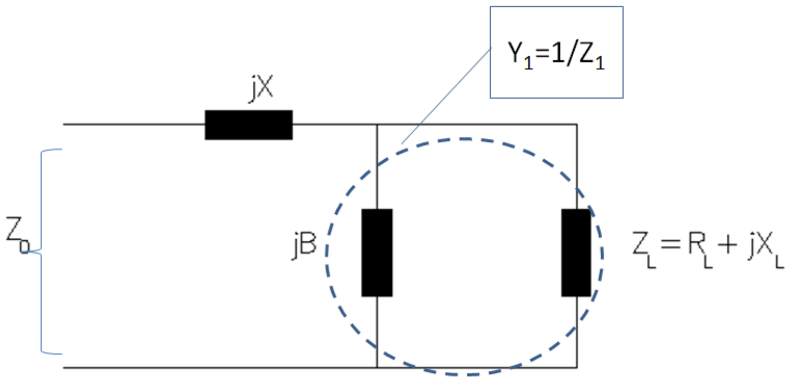
- From the scale “SWR” read SWR ≈ 1,82
- From the scale “dBs” read SWR in dB ≈ 5,2dB
- From the scale “RTN LOSS(dB)” read R in dB ≈ -10,7dB
- From the scale “RFL COEFF,P” read power reflection coefficient $|s|^2$≈ 0,082
- From the scale “RFL COEFF,E or I” read reflection coefficient for voltage or current $|\Gamma_L|$= $|\Gamma_G|$ ≈ 0,285
- From the circular scale “Angle of reflection coefficient” read in the intersection of lines drawn in steps 3 and 5 and this scale the corresponding angles of reflection coefficient , angle of $\Gamma_L$≈73° and angle of $\Gamma_G$≈16,6°

## 4. Distance to first voltage maximum and minimum

## 5. Impedance matching

### 5.1 Shunt-series L-Network, by calculation

Load impedance $Z_0 = R_L+jX_L$ shall be matched to system impedance $Z_0$ to maximize power transfer from generator to load. Matching L- network consists of shunt susceptance jB and series reactance jX .




**Kuva 5:L-network matching**

Calculate first impedance $Z_1$, Load $Z_L$ in parallel with jB

$$Z_1 = \frac{\frac{1}{jB}(R_L + jX_L)}{\frac{1}{jB} + (R_L + jX_L)} = \frac{(R_L + jX_L)}{1 - BX_L + jR_L B}$$

Develop $Z_1$ further to separate its real and imaginary parts:

- Multiply numerator and denominator of $Z_1$ with $(1-BX_L-jR_LB)$->

$$Z_1 = \frac{(R_L + jX_L)(1 - BX_L - jR_L B)}{(1 - X_L B)^2 + {R_L}^2 B^2} = \frac{R_L \cancel{- R_L B X_L + R_L B X_L} + j\left(X_L - B{X_L}^2 - {R_L}^2 B\right)}{(1 - X_L B)^2 + {R_L}^2 B^2}$$

$$= \frac{R_L + j\left(X_L - B{X_L}^2 - {R_L}^2 B\right)}{1 + {X_L}^2 B^2 - 2X_L B + {R_L}^2 B^2} = \frac{R_L + j\left(X_L - B{X_L}^2 - {R_L}^2 B\right)}{\left({R_L}^2 + {X_L}^2\right)B^2 - 2X_L B + 1}$$

Real part of $Z_1$ has to be equal to the system impedance $Z_0$ -> Solve B to fulfill that condition

$$\frac{R_L}{\left({R_L}^2 + {X_L}^2\right)B^2 - 2X_L B + 1} = Z_0$$

$$\left({R_L}^2 + {X_L}^2\right)B^2 - 2X_L B + 1 = \frac{R_L}{Z_0} \rightarrow$$

$$Z_0\left({R_L}^2 + {X_L}^2\right)B^2 - 2Z_0 X_L B + (Z_0 - R_L) = 0$$

This is a quadratic equation which roots are

$$B = \frac{2Z_0X_L \pm \sqrt{(2Z_0X_L)^2 - 4Z_0\left({R_L}^2 + {X_L}^2\right)(Z_0 - R_L)}}{2Z_0\left({R_L}^2 + {X_L}^2\right)}$$

$$= \frac{Z_0X_L \pm \sqrt{{Z_0}^2{X_L}^2 - \left({R_L}^2 + {X_L}^2\right)\left({Z_0}^2 - Z_0R_L\right)}}{Z_0\left({R_L}^2 + {X_L}^2\right)}$$

$$= \frac{Z_0X_L \pm \sqrt{{Z_0}^2{X_L}^2 - {R_L}^2{Z_0}^2 + {R_L}^3 Z_0 - {Z_0}^2{X_L}^2 + {X_L}^2 R_L Z_0}}{Z_0\left({R_L}^2 + {X_L}^2\right)}$$

$$= \frac{Z_0X_L \pm \sqrt{Z_0({X_L}^2Z_0) - Z_0({R_L}^2Z_0) + Z_0{R_L}^3 - Z_0({X_L}^2Z_0) + Z_0({X_L}^2R_L)}}{Z_0\left({R_L}^2 + {X_L}^2\right)}$$

$$= \frac{Z_0X_L \pm \sqrt{Z_0\left(\cancel{{X_L}^2Z_0} - {R_L}^2Z_0 + {R_L}^3 \cancel{- {X_L}^2Z_0} + {X_L}^2R_L\right)}}{Z_0\left({R_L}^2 + {X_L}^2\right)}$$

$$= \frac{Z_0X_L \pm \sqrt{Z_0\left(-{R_L}^2Z_0 + {R_L}^3 + {X_L}^2R_L\right)}}{Z_0\left({R_L}^2 + {X_L}^2\right)}$$

$$= \frac{\frac{Z_0X_L}{Z_0} \pm \sqrt{\left(\frac{1}{Z_0}\right)^2 Z_0\left(-{R_L}^2Z_0 + {R_L}^3 + {X_L}^2R_L\right)}}{\frac{Z_0\left({R_L}^2 + {X_L}^2\right)}{Z_0}}$$

$$= \frac{X_L \pm \sqrt{\frac{1}{Z_0}R_L\left({R_L}^2 + {X_L}^2 - R_LZ_0\right)}}{\left({R_L}^2 + {X_L}^2\right)} \rightarrow$$

$$\boxed{B = \frac{X_L \pm \sqrt{\frac{R_L}{Z_0}}\sqrt{\left({R_L}^2 + {X_L}^2 - R_LZ_0\right)}}{\left({R_L}^2 + {X_L}^2\right)}}$$

**Kaava 2**

Series reactance jX has to compensate the imaginary part of $Z_1$

$$\frac{\left(X_L - B{X_L}^2 - {R_L}^2B\right)}{\left({R_L}^2 + {X_L}^2\right)B^2 - 2X_LB + 1} = -X$$

and because

$$\frac{R_L}{\left({R_L}^2 + {X_L}^2\right)B^2 - 2X_LB + 1} = \frac{R_L}{\frac{R_L}{Z_0}} = Z_0$$

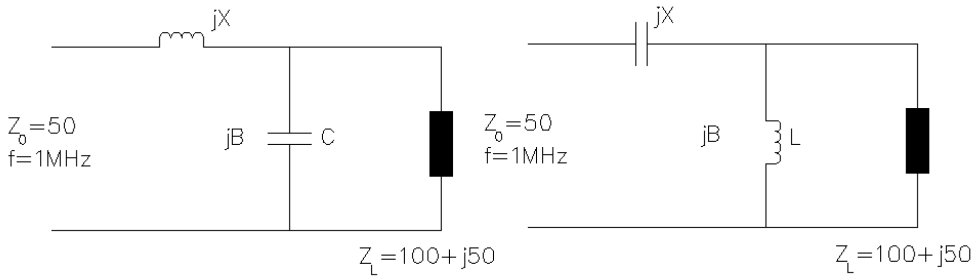
$$\frac{\left(X_L - B{X_L}^2 - {R_L}^2B\right)}{\frac{R_L}{Z_0}} = -X$$

$$-X = \frac{X_l - \left({X_L}^2 + {R_L}^2\right)B}{\frac{R_L}{Z_0}} = \frac{-Z_0X_L + BZ_0\left({X_L}^2 + {R_L}^2\right)}{R_L} \rightarrow$$

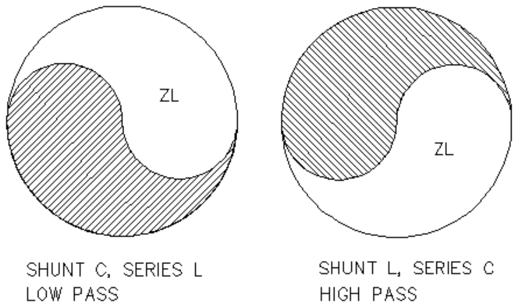
$$\boxed{X = \frac{BZ_0\left({R_L}^2 + {X_L}^2\right) - Z_0X_L}{R_L}}$$

**Kaava 3**

### 5.1.1 Example: Match load



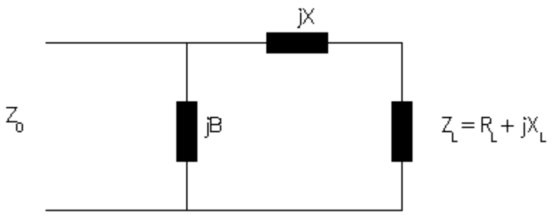

| Shunt C, series L | Shunt L, series C |
|---|---|
| $B = \frac{X_L + \sqrt{\frac{R_L}{Z_0}}\sqrt{(R_L{}^2 + X_L{}^2 - R_L Z_0)}}{(R_L{}^2 + X_L{}^2)} = \frac{50 + \sqrt{\frac{100}{50}}\sqrt{100^2 + 50^2 - 100 * 50}}{100^2 + 50^2} = 0{,}013798S = 72{,}47\Omega\ (capacitor)$ | $B = \frac{X_L - \sqrt{\frac{R_L}{Z_0}}\sqrt{(R_L{}^2 + X_L{}^2 - R_L Z_0)}}{(R_L{}^2 + X_L{}^2)} = \frac{50 - \sqrt{\frac{100}{50}}\sqrt{100^2 + 50^2 - 100 * 50}}{100^2 + 50^2} -0{,}005798S = -172{,}47\Omega\ (inductor)$ |
| $Y_L = \frac{1}{100 + j50} = 0{,}008 - j0{,}004S$ | $Y_L = \frac{1}{100 + j50} = 0{,}008 - j0{,}004S$ |
| $Y_1 = 0{,}008 - j0{,}004 + j0{,}013798 = 0{,}008 + j0{,}009798S$ | $Y_1 = 0{,}008 - j0{,}004 - j0{,}005798 = 0{,}008 - j0{,}009798S$ |
| $Z_1 = \frac{1}{Y_1} = \frac{1}{0{,}008 + j0{,}009798} = 50{,}00 - 61{,}23\Omega$ | $Z_1 = \frac{1}{Y_1} = \frac{1}{0{,}008 - j0{,}009798} = 50{,}00 + 61{,}23\Omega$ |
| $\boldsymbol{C} = \frac{1}{72{,}47 * 2\pi * 1 * 10^6}F = \boldsymbol{2{,}197nF}$ | $\boldsymbol{L} = \frac{172{,}47}{2\pi * 1 * 10^6}H = \boldsymbol{27{,}4uH}$ |
| $X = \frac{BZ_0(R_L{}^2 + X_L{}^2) - Z_0 X_L}{R_L} = \frac{0{,}013798 * 50(100^2 + 50^2) - 50 * 50}{100} = 61{,}23\Omega\ (inductor)$ | $X = \frac{BZ_0(R_L{}^2 + X_L{}^2) - Z_0 X_L}{R_L} = \frac{-0{,}005798S * 50(100^2 + 50^2) - 50 * 50}{100} = -61{,}23\Omega\ (capacitor)$ |
| $\boldsymbol{L} = \frac{61{,}23}{2\pi * 1 * 10^6}H = \boldsymbol{9{,}75uH}$ | $\boldsymbol{C} = \frac{1}{61{,}23 * 2\pi * 1 * 10^6}F = \boldsymbol{2{,}600nF}$ |




Kuva 6: Allowable Load impedance area for shunt-series network in Smith Chart representation

## 5.2 Series shunt- L-Network, by calculation

Matching L- network consists of a series reactance jX and a shunt susceptance jB.




Calculate first impedance $Z_1$, Load $Z_L$ in series with jX

$$Z_1 = jX + R_L + jX_L = j(X_L + X)$$

$$Y_1 = \frac{1}{Z_1} = \frac{1}{R_L + j(X_L + X)}$$

Develop Y1 to separate its real and imaginary parts, multiply numerator and denominator by $\mathrm{R_L - j(X_L + X)}$

$$Y_1 = \frac{R_L - j(X_L + X)}{{R_L}^2 - \left(j(X_L + X)\right)^2} = \frac{R_L - j(X_L + X)}{{R_L}^2 + {X_L}^2 + X^2 + 2X_L X}$$

Real part of admittance $Y_1$ must be equal to $1/Z_0$ ->

$$\frac{R_L}{{R_L}^2 + {X_L}^2 + X^2 + 2X_L X} = \frac{1}{Z_0} \rightarrow$$

$$X^2 + 2X_L X + {R_L}^2 + {X_L}^2 - R_L Z_0 = 0 \rightarrow$$

$$X = \frac{-2X_L \pm \sqrt{(2X_L)^2 - 4\left({R_L}^2 + {X_L}^2 - R_L Z_0\right)}}{2} \rightarrow$$

$$\boxed{X = -X_L \pm \sqrt{R_L(Z_0 - R_L)}}$$

**Kaava 4**

Imaginary part of admittance $Y_1$ has to compensated by B, and

$$\frac{R_L}{{R_L}^2 + {X_L}^2 + X^2 + 2X_L X} = \frac{1}{Z_0} = \frac{R_L}{R_L Z_0} \rightarrow$$

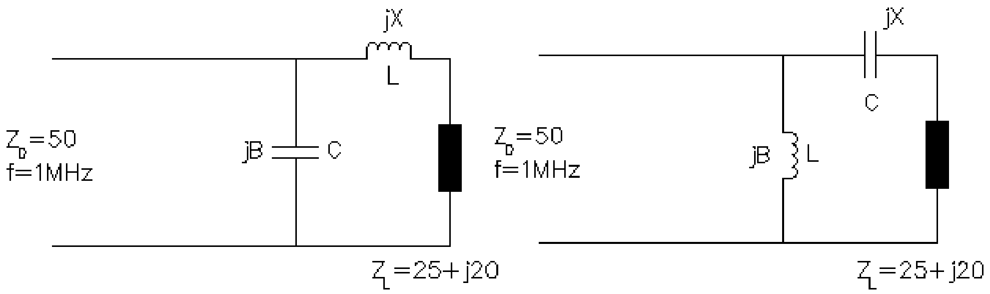
$$\frac{X_L + X}{R_L Z_0} = \frac{\cancel{X_L - X_L} \pm \sqrt{R_L(Z_0 - R_L)}}{R_L Z_0} = \frac{\frac{1}{R_L} \pm \sqrt{R_L(Z_0 - R_L)}}{\frac{1}{\cancel{R_L}} \cancel{R_L} Z_0} = -B \rightarrow$$

$$\boxed{B = \frac{\pm\sqrt{\frac{(Z_0 - R_L)}{R_L}}}{Z_0}}$$

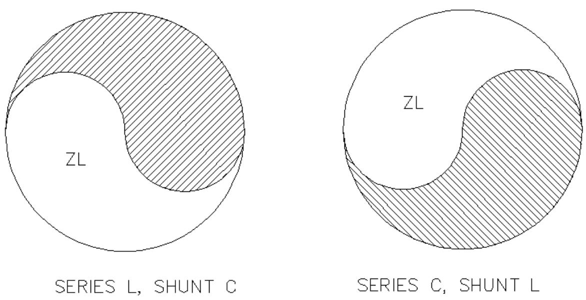
**Kaava 5**

### 5.2.1 Example: Match load

L-network match, series- shunt

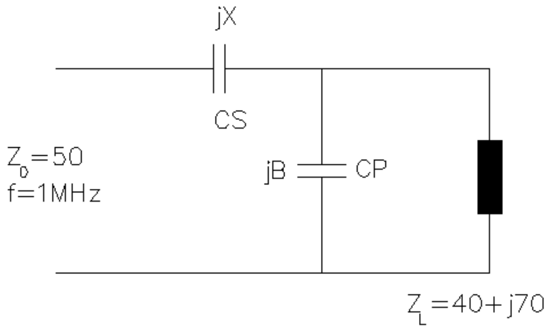


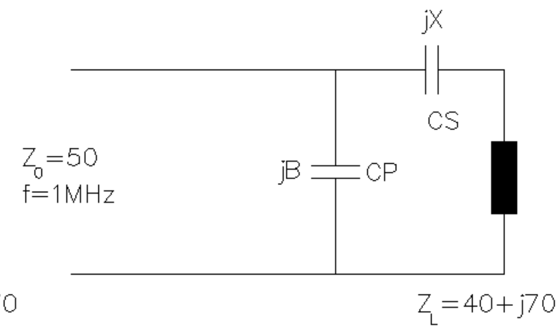

| Series L, Shunt C | Series C, Shunt L |
|---|---|
| $X = -X_L + \sqrt{R_L(Z_0 - R_L)}$ $= -(20) + \sqrt{25(50-25)}$ $= 5{,}00\Omega$ | $X = -X_L - \sqrt{R_L(Z_0 - R_L)}$ $= -(20) - \sqrt{25(50-25)}$ $= -45{,}00\Omega$ |
| $B = \frac{+\sqrt{\frac{(Z_0 - R_L)}{R_L}}}{Z_0} = \frac{+\sqrt{\frac{(50-25)}{25}}}{50} = 0{,}020S$ | $B = \frac{-\sqrt{\frac{(Z_0 - R_L)}{R_L}}}{Z_0} = \frac{-\sqrt{\frac{(50-25)}{25}}}{50} = -0{,}020S$ |
| $\boldsymbol{C} = \frac{0{,}020}{2\pi * 1 * 10^6} = \mathbf{3{,}18}\boldsymbol{nF}$ | $\boldsymbol{L} = \frac{1/0{,}020}{2\pi * 1 * 10^6} = \mathbf{7{,}96}\boldsymbol{uH}$ |
| $\boldsymbol{L} = \frac{5{,}00}{2\pi * 1 * 10^6} = \mathbf{796}\boldsymbol{nH}$ | $\boldsymbol{C} = \frac{1}{45{,}00 * 2\pi * 1 * 10^6} = \mathbf{3{,}53}\boldsymbol{nF}$ |



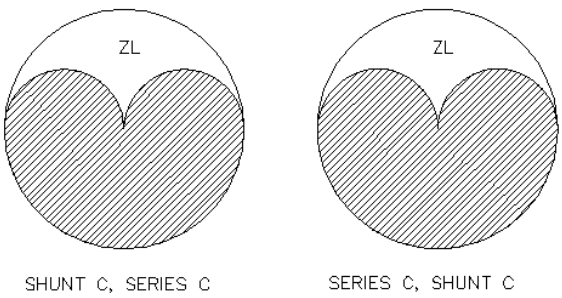

**Kuva 7: Allowable Load impedance area for series -shunt -network in Smith Chart representation**

### 5.2.2 Example: Match load



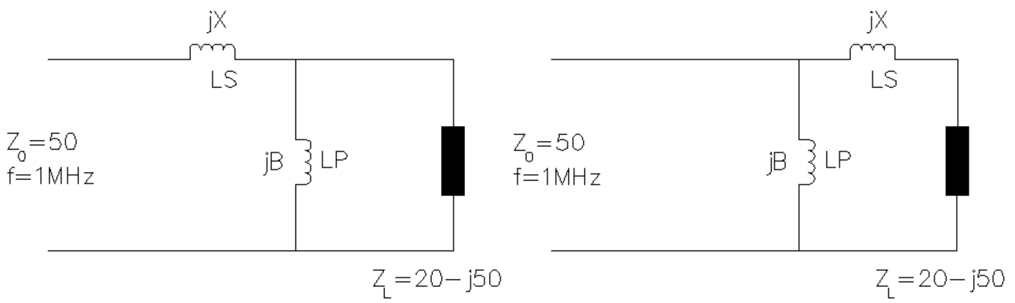




| **Shunt C, series C** | **Series C, shunt C** |
|---|---|
| $B=\dfrac{X_L \pm \sqrt{\frac{R_L}{Z_0}}\sqrt{\left({R_L}^2+{X_L}^2-R_L Z_0\right)}}{\left({R_L}^2+{X_L}^2\right)}$ $=\dfrac{70 \pm \sqrt{\frac{40}{50}}\sqrt{(40^2+70^2-40*50)}}{(40^2+70^2)}$ $=0{,}020S\ (not\ applicable, would\ require\ L\ in\ series)$ $=0{,}001538S$ | $X=-X_L \pm \sqrt{R_L(Z_0-R_L)}$ $=-70 \pm \sqrt{40(50-40)}$ $=-50{,}00\Omega$ $=-90{,}00\Omega\ (not\ applicable, would\ require\ L\ in\ parallel)$ |
| $\boldsymbol{CP}=\dfrac{0{,}001531}{2\pi*1*10^6}F=\mathbf{244,9}\boldsymbol{pF}$ | $\boldsymbol{CS}=\dfrac{1}{50{,}00*2\pi*1*10^6}F=\mathbf{3,18}\boldsymbol{nF}$ |
| $X=\dfrac{BZ_0\left({R_L}^2+{X_L}^2\right)-Z_0X_L}{R_L}$ $=\dfrac{0{,}001538*50(40^2+70^2)-50*70}{40}$ $=-75{,}00\Omega\ (capacitor)$ | $B=\dfrac{\pm\sqrt{\frac{(Z_0-R_L)}{R_L}}}{Z_0}=\dfrac{\pm\sqrt{\frac{(50-40)}{40}}}{50}$ $=0{,}010S\ (capacitor)$ $=-0{,}010S\ (inductor, not\ applicable)$ |
| $\boldsymbol{CS}=\dfrac{1}{75{,}00*2\pi*1*10^6}F=\mathbf{2,12}\boldsymbol{nF}$ | $\boldsymbol{CP}=\dfrac{0{,}010}{2\pi*1*10^6}F=\mathbf{1,59}\boldsymbol{nF}$ |

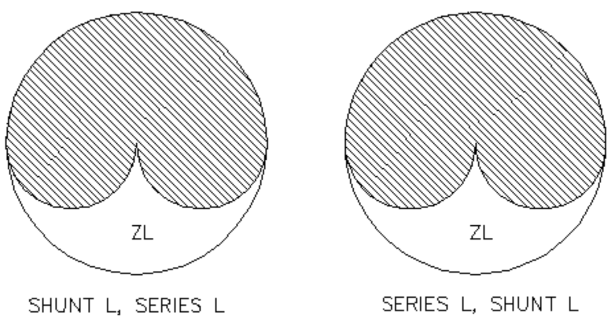



**Kuva 8 Allowable Load impedance area for capacitors only-network in Smith Chart representation**

### 5.2.3 Example: Match load



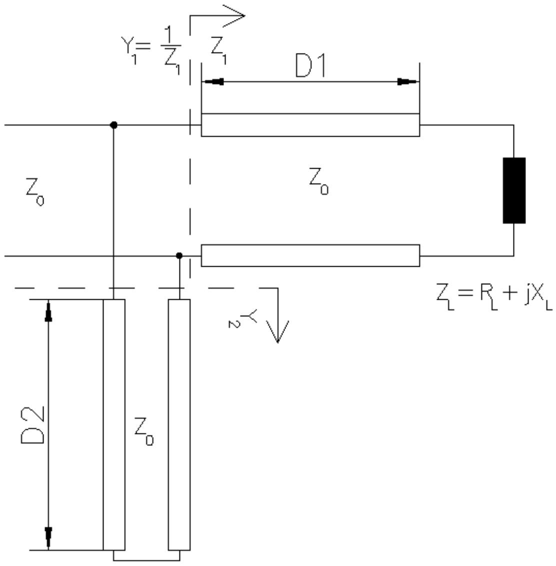

| Shunt L, series L | Series L, shunt L |
|---|---|
| $B = \frac{X_L \pm \sqrt{\frac{R_L}{Z_0}}\sqrt{(R_L{}^2 + X_L{}^2 - R_L Z_0)}}{(R_L{}^2 + X_L{}^2)} = \frac{-50 \pm \sqrt{\frac{20}{50}}\sqrt{(20^2 + (-50)^2 - 20 * 50)}}{(20^2 + (-50)^2)}$ $= -0{,}00773S$ $= -0{,}02674S\ (not\ applicable, would\ require\ series\ C\ )$ | $X = -X_L \pm \sqrt{R_L(Z_0 - R_L)} = -(-50) \pm \sqrt{20(50-20)}$ $= 74{,}49\Omega\ (not\ applicable, would\ require\ shunt\ C$ $= 25{,}51\Omega$ |
| $\boldsymbol{LP} = \frac{1}{0{,}00773 * 2\pi * 1 * 10^6} H = \boldsymbol{20{,}6uH}$ | $\boldsymbol{LS} = \frac{25{,}51}{2\pi * 1 * 10^6} H = \boldsymbol{4{,}06uH}$ |
| $X = \frac{BZ_0(R_L{}^2 + X_L{}^2) - Z_0 X_L}{R_L} = \frac{-0{,}00773 * 50(20^2 + (-50)^2) - 50 * (-50)}{20}$ $= 68{,}96\Omega$ | $B = \frac{\pm\sqrt{\frac{(Z_0 - R_L)}{R_L}}}{Z_0} = \frac{\pm\sqrt{\frac{(50-20)}{20}}}{50}$ $= 0{,}0245S\ (capacitor. not\ applicable)$ $= -0{,}0245S\ (inductor)$ |
| $\boldsymbol{LS} = \frac{68{,}96}{2\pi * 1 * 10^6} H = \boldsymbol{10{,}98uH}$ | $\boldsymbol{LP} = \frac{1}{0{,}0245 * 2\pi * 1 * 10^6} H = \boldsymbol{6{,}50uH}$ |




**Kuva 9 Allowable Load impedance area for inductors only-network in Smith Chart representation**

## 5.3 Shorted parallel stub as tuning element

Load matching can be achieved by paralleling a length of a short circuited transmission line 2 (length D2) placed at a proper distance (transmission line length D1) from the load. Both transmission lines here are assumed to be lossless and having a characteristic impedance $Z_0$.




Impedance towards the load as seen in front of transmission line 1 is

$$Z_1 = Z_0 \frac{Z_L + jZ_0 tan\beta D1}{Z_0 + jZ_L tan\beta D1} = \frac{Z_0[(R_L + jX_L) + jZ_0 tan\beta D1]}{Z_0 + j[(R_L + jX_L)tan\beta D1]} = \frac{Z_0 R_L + jZ_0 X_L + jZ_0{}^2 tan\beta D1}{Z_0 - X_L tan\beta D1 + jR_L tan\beta D1}$$

$$Y_1 = \frac{Z_0 - X_L tan\beta D1 + jR_L tan\beta D1}{Z_0 R_L + jZ_0 X_L + jZ_0{}^2 tan\beta D1} = \frac{Z_0 - X_L tan\beta D1 + jR_L tan\beta D1}{Z_0 R_L + j(Z_0 X_L + Z_0{}^2 tan\beta D1)}$$

Develop $Y_1$ further to separate its real and imaginary parts: multiply numerator and denominator by $(Z_0 R_L - j(Z_0 X_L + Z_0{}^2 tan\beta D1))$

$$Y_1 = \frac{(Z_0 - X_L tan\beta D1 + jR_L tan\beta D1)(Z_0 R_L - jZ_0 X_L - jZ_0{}^2 tan\beta D1)}{Z_0{}^2 R_L{}^2 + Z_0{}^2 X_L{}^2 + 2Z_0{}^3 X_L tan\beta D1 + Z_0{}^4 tan^2\beta D1}$$

$$= \frac{Z_0{}^2 R_L - jZ_0{}^2 X_L - jZ_0{}^3 tan\beta D1 \cancel{- Z_0 R_L X_L tan\beta D1} + jZ_0 X_L{}^2 tan\beta D1 + jZ_0{}^2 X_L tan^2\beta D1 + jZ_0 R_L{}^2 tan\beta D1 \cancel{+ Z_0 R_L X_L tan\beta D1} + Z_0{}^2 R_L tan^2\beta D1}{Z_0{}^2(R_L{}^2 + X_L{}^2 + 2Z_0 X_L tan\beta D1 + Z_0{}^2 tan^2\beta D1}$$

$$Y_1 = \frac{\cancel{Z_0{}^2} R_L(1 + tan^2\beta D1) + j\cancel{Z_0{}^2}\left(-X_L - Z_0 tan\beta D1 + \frac{X_L{}^2}{Z_0} tan\beta D1 + X_L tan^2\beta D1 + \frac{R_L{}^2}{Z_0} tan\beta D1\right)}{\cancel{Z_0{}^2}(R_L{}^2 + X_L{}^2 + 2Z_0 X_L tan\beta D1 + Z_0{}^2 tan^2\beta D1)}$$

real part of admittance $Y_1$ has to be equal to $1/Z_0$ , because stub to be calculated later will compensate the imaginary part->

$$\frac{R_L(1+tan^2\beta D1)}{{R_L}^2+{X_L}^2+2Z_0X_Ltan\beta D1+{Z_0}^2tan^2\beta D1}=\frac{1}{Z_0}\rightarrow$$

$${Z_0}^2tan^2\beta D1\,{R_L}^2-R_LZ_0tan^2\beta D1+2Z_0X_Ltan\beta D1+{X_L}^2-R_LZ_0=0\rightarrow$$

$$({Z_0}^2-R_LZ_0)tan^2\beta D1+2Z_0X_Ltan\beta D1+{R_L}^2+{X_L}^2-R_LZ_0=0\rightarrow$$

$$\boxed{\beta D1=atan\left(\frac{-2Z_0X_L\pm\sqrt{(2Z_0X_L)^2-4({Z_0}^2-R_LZ_0)({R_L}^2+{X_L}^2-R_LZ_0)}}{2({Z_0}^2-R_LZ_0)}\right)}$$

**Kaava 6: Length of transmission line in shorted stub matching**

Only the positive root is meaningful.

Length of transmission line D1 is

$$\boxed{D1=\frac{\beta D1}{2\pi}\lambda}$$

Impedance $Z_2$ of the shorted stub of length D2 is, because $Z_{L,stub}$ =0

$$Z_2=Z_0\frac{jZ_0tan\beta D1}{Z_0}=jZ_0tan\beta D20->Y_2=\frac{1}{jZ_0tan\beta D2},$$

therefore

$$\frac{j(-X_L-Z_0tan\beta D1+\frac{{X_L}^2}{Z_0}tan\beta D1+X_Ltan^2\beta D1+\frac{{R_L}^2}{Z_0}tan\beta D1)}{R_L(1+tan^2\beta D1)Z_0}=\frac{1}{-jZ_0tan\beta D2}\rightarrow$$

$$\boxed{\beta D2=atan\left(\frac{R_L(1+tan^2\beta D1)}{tan\beta D1\left(-Z_0+\frac{{X_L}^2}{Z_0}+X_Ltan\beta D1+\frac{{R_L}^2}{Z_0}\right)-X_L}\right)}$$

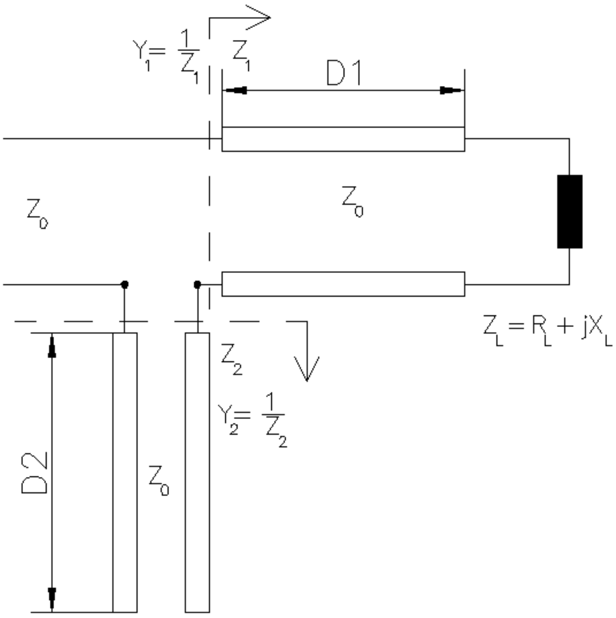
**Kaava 7: Length of shorted stub in shorted stub matching**

Length of stub D2 is

$$\boxed{D2=\frac{\beta D2}{2\pi}\lambda}$$

## 5.4 Open series stub as tuning element

Load matching can be achieved by a length of a open transmission line 2 (length D2) placed in series at a proper distance (transmission line length D1) from the load. Both transmission lines here are assumed to be lossless and having a characteristic impedance $Z_0$.






P.Viljakainen
January 2021

## 6. Lossy transmission line terminated with an arbitrary impedance

Wave functions for voltage and current as function of distance z (z=0 at generator side of the line) for a lossy transmission line are

$$V(z) = V^{+}e^{(-\gamma z)} + V^{-}e^{(\gamma z)}$$

$$I(z) = \frac{V^{+}}{Z_0}e^{(-\gamma z)} - \frac{V^{-}}{Z_0}e^{(\gamma z)}$$

Where

$$\boxed{\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}}$$

**Kaava 8:Propagation constant**

$\gamma$ =Propagation constant $[\gamma] = \frac{1}{m}$

α = Attenuation constant$[\alpha] = \frac{1}{m}$; α/dB=20 log $e^{\alpha}$ =20α log e = 8,686 α

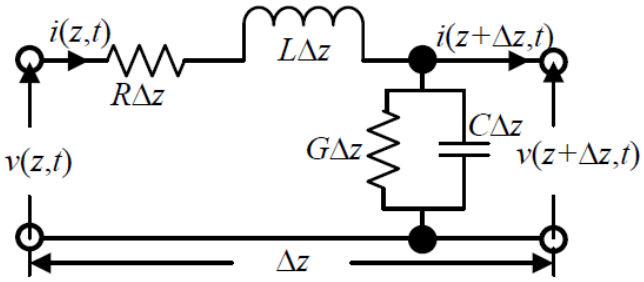
β = Phase constant $[\beta] = \frac{rad}{m}$

R, L, G, C = Transmission line distributed constants

The velocity of propagation or the phase velocity is given by

$$\boxed{u_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}}$$

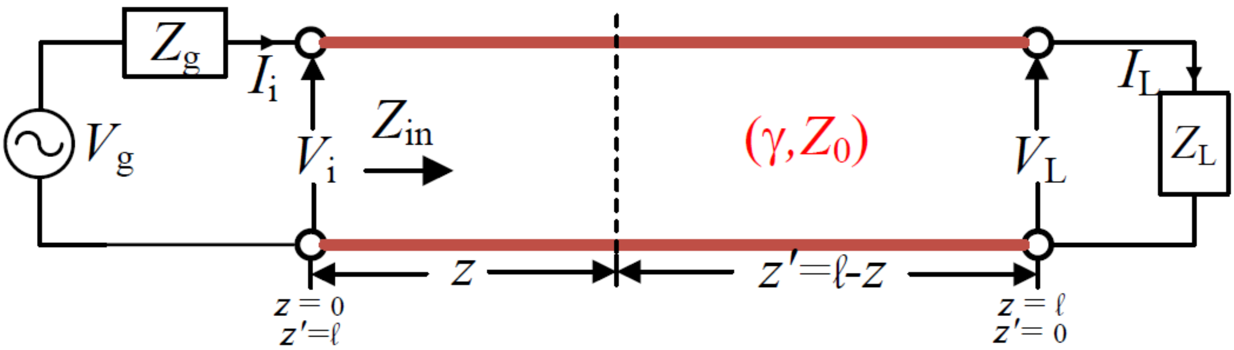
**Kaava 9: Speed of propagation**




and the characteristic impedance $Z_0$ is

$$\boxed{Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}}$$

**Kaava 10:Characteristic impedance of a lossy transmission line**




$$Z_L = \frac{V_L}{I_L} = \frac{V^+e^{(-\gamma l)} + V^-e^{(\gamma l)}}{\frac{V^+}{Z_0}e^{(-\gamma l)} - \frac{V^-}{Z_0}e^{(\gamma l)}}$$

Because

$$V_L = V(l) = V^+e^{(-\gamma l)} + V^-e^{(\gamma l)} \rightarrow V^+e^{(-\gamma l)} = V_L - V^-e^{(\gamma l)}$$

$$I_L = I(l) = \frac{V^+}{Z_0} + e^{(-\gamma l)} - \frac{V^-}{Z_0}e^{(\gamma l)}$$

$V^-$ and $V^+$ can be solved

$$I_L = \frac{V_L - V^-e^{(\gamma l)}}{Z_0} - \frac{V^-e^{(\gamma l)}}{Z_0}$$

$$\rightarrow V^- = \frac{1}{2}(V_L - I_L Z_0)e^{(-\gamma l)} = \frac{I_L}{2}(Z_L - Z_0)e^{(-\gamma l)}$$

$$V_L = V^+e^{(-\gamma l)} + V^-e^{(\gamma l)} = V^+e^{(-\gamma l)} + \frac{1}{2}(V_L - I_L Z_0)\, e^{(-\gamma l)}e^{(-\gamma l)}$$

$$\rightarrow V^{+} = \frac{1}{2}(V_L + I_L Z_0)e^{(\gamma l)} = \frac{I_L}{2}(Z_L + Z_0)e^{(\gamma l)}$$

and V(z) and I(z) now expressed as

$$V(z) = V^{+}e^{(-\gamma z)} + V^{-}e^{(\gamma z)}$$

$$V(z) = \frac{I_L}{2}\left[(Z_L + Z_0)e^{\gamma(l-z)} + (Z_L - Z_0)e^{-\gamma(l-z)}\right]$$

$$I(z) = \frac{V^{+}}{Z_0}e^{(-\gamma z)} - \frac{V^{-}}{Z_0}e^{(\gamma z)}$$

$$I(z) = \frac{I_L}{2Z_0}\left[(Z_L + Z_0)e^{\gamma(l-z)} - (Z_L - Z_0)e^{-\gamma(l-z)}\right]$$

and by assigning z' = l-z;

$$V(z) = \frac{I_L}{2}[(Z_L + Z_0)e^{\gamma z\prime} + (Z_L - Z_0)e^{-\gamma z\prime}]$$

$$I(z) = \frac{I_L}{2Z_0}[(Z_L + Z_0)e^{\gamma z\prime} - (Z_L - Z_0)e^{-\gamma z\prime}]$$

These functions can be written as

$$V(z') = \frac{I_L}{2}(Z_L + Z_0)e^{\gamma z'} + \frac{I_L}{2}(Z_L - Z_0)e^{-\gamma z'} = \frac{I_L Z_L e^{\gamma z'} + I_L Z_0 e^{\gamma z'}}{2} + \frac{I_L Z_L e^{-\gamma z'} - I_L Z_0 e^{-\gamma z'}}{2}$$

$$= \frac{I_L Z_L e^{\gamma z'}}{2} + \frac{I_L Z_L e^{-\gamma z'}}{2} + \frac{I_L Z_0 e^{\gamma z'}}{2} - \frac{I_L Z_0 e^{-\gamma z'}}{2}$$

$$V(z') = I_L Z_L\left(\frac{e^{\gamma z'} + e^{-\gamma z'}}{2}\right) + I_L Z_0\left(\frac{e^{\gamma z'} - e^{-\gamma z'}}{2}\right)$$

Because

$$\left(\frac{e^{\gamma z'} + e^{-\gamma z'}}{2}\right) = \cosh(\gamma z')$$

and

$$\left(\frac{e^{\gamma z'} - e^{-\gamma z'}}{2}\right) = \sinh(\gamma z')$$

$$\boxed{V(z') = I_L[Z_L \cosh(\gamma z') + Z_0 \sinh(\gamma z')]}$$

**Kaava 11: Voltage of a lossy transmission line as function of distance**

$$I(z) = \frac{I_L}{2Z_0}[(Z_L + Z_0)e^{\gamma z\prime} - (Z_L - Z_0)e^{-\gamma z\prime}] = \frac{I_L Z_L}{Z_0}\left(\frac{e^{\gamma z'} - e^{-\gamma z'}}{2}\right) + \frac{I_L Z_L}{Z_0}\left(\frac{e^{\gamma z'} + e^{-\gamma z'}}{2}\right)$$

$$\boxed{I(z') = \frac{I_L}{Z_0}[Z_L \sinh(\gamma z') + Z_0 \sinh(\gamma z')]}$$

**Kaava 12: Current of a lossy transmission line as function of distance**

Load impedance at distance z' from the load is

$$Z(z') = \frac{V_L}{I_L} = \frac{I_L[Z_L \cosh(\gamma z') + Z_0 \sinh(\gamma z')]}{\frac{I_L}{Z_0}[Z_L \sinh(\gamma z') + Z_0 \sinh(\gamma z')]} = Z_0 \frac{Z_L \cosh(\gamma z') + Z_0 \sinh(\gamma z')}{Z_L \sinh(\gamma z') + Z_0 \sinh(\gamma z')}$$

divide numerator and denominator by cosh $(\gamma z')$->

$$Z(z') = Z_0 \frac{Z_L \frac{\cosh(\gamma z')}{\cosh(\gamma z')} + Z_0 \frac{\sinh(\gamma z')}{\cosh(\gamma z')}}{Z_L \frac{\sinh(\gamma z')}{\cosh(\gamma z')} + Z_0 \frac{\cosh(\gamma z')}{\cosh(\gamma z')}}$$

and when

$$\frac{\sinh(\gamma z')}{\cosh(\gamma z')} = \tanh(\gamma z')$$

$$Z(z') = Z_0 \frac{Z_L + Z_0 \tanh(\gamma z')}{Z_0 + Z_L \tanh(\gamma z')}$$

at generator, when z'=l

$$\boxed{Z(z' = l) = Z_{IN} = Z_0 \frac{Z_L + Z_0 \tanh(\gamma l)}{Z_0 + Z_L \tanh(\gamma l)}}$$

**Kaava 13:Input impdance of a lossy transmission line**

Furthermore, as $\gamma = \alpha + j\beta$ and

$$\tanh(x + jy) = \frac{\tanh(x) + j \tan(y)}{1 + j\, tanh(x) \tan(y)}$$

$$\boxed{Z_{IN} = Z_0 \frac{Z_L + Z_0 \left(\frac{\tanh(\alpha l) + j \tan(\beta l)}{1 + j\, tanh(\alpha l) \tan(\beta l)}\right)}{Z_0 + Z_L \left(\frac{\tanh(\alpha l) + j \tan(\beta l)}{1 + j\, tanh(\alpha l) \tan(\beta l)}\right)}}$$

Power supplied by the generator is

$$\boxed{P_{avG} = \frac{1}{2} Re[V_G I_{IN}{}^*]}$$

**Kaava 14:Power supplied by the generator**

Power delivered to the load is

$$\boxed{P_{avL} = \frac{1}{2} Re[V_L I_L{}^*] = \frac{1}{2} \left|\frac{V_L}{I_L}\right|^2 R_L = \frac{1}{2} |I_L|^2 R_L}$$

**Kaava 15: Power transmitted to load**

Where

$V_G$, $I_{IN}$, $V_L$ and $I_L$ are the peak values of voltage and current.

Voltage reflection coefficient

$$\begin{cases} \Gamma(z) = \dfrac{V^- e^{\gamma z}}{V^+ e^{-\gamma z}} = \dfrac{V^-}{V^+} e^{2\gamma z} \\ V(z) = V^+ e^{(-\gamma z)} + V^- e^{(\gamma z)} \end{cases}$$

$$V(z) = V^+ e^{(-\gamma z)} [1 + \Gamma(z)]$$

$$I(z) = \frac{V^+}{Z_0} e^{(-\gamma z)} [1 + \Gamma(z)]$$

$$Z(z) = \frac{V(z)}{I(z)} = Z_0 \frac{1 + \Gamma(z)}{1 - \Gamma(z)}$$

At load (z=l)

$$\boxed{\Gamma(l) = \Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = |\Gamma_L| e^{j\theta_\gamma}}$$

**Kaava 16:Voltage reflection coefficient at load**

$$\Gamma(z) = \frac{V^-}{V^+} e^{2\gamma z} \rightarrow \frac{V^-}{V^+} = \Gamma(l) e^{-2\gamma l} = \Gamma_L e^{-2\gamma l}$$

At distance z

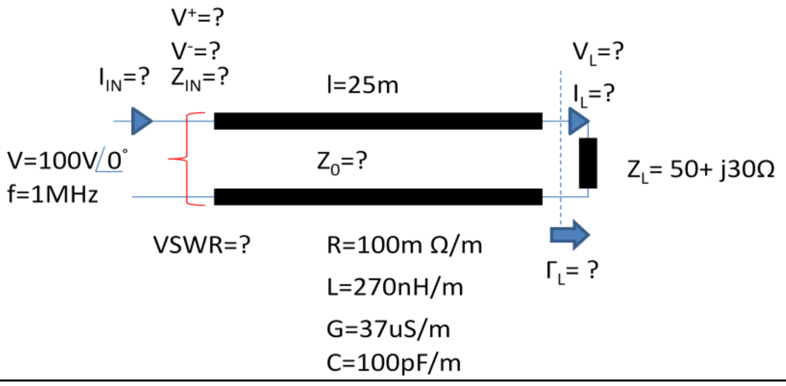
$$\Gamma(z) = \Gamma_L e^{-2\gamma l} e^{2\gamma z} = \Gamma_L e^{2\gamma(z-l)}$$

Voltage standing wave ratio is defined as

$$\boxed{VSWR = \frac{|V_{max}|}{|V_{min}|} = S = \frac{1 + |\Gamma|}{1 - |\Gamma|}}$$

**Kaava 17:Voltage standing wave ratio**

## 6.1 Example:



Calculate:

a) Transmission line propagation constant $\gamma$
b) Velocity of propagation in the transmission line
c) Input impedance $Z_{IN}$
d) Input current $I_{IN}$
e) Characteristic impedance of transmission line $Z_0$
f) Voltage at load $V_L$
g) Load current $I_L$
h) Transmission line voltage at 10m from the load
i) Power delivered by the generator $P_{avG}$
j) Power delivered to the load $P_{avL}$
k) Voltage reflection coefficient at load $\Gamma_L$
l) Voltage standing wave ratio VSWR

$R = 100x10^{-3}\Omega/m$
$L = 270x10^{-9}H/m; j2\pi 1x10^6 x270x10^{-9} = j1{,}696\ \Omega/m$
$G = 37x10^{-6}\ S/m$
$C = 100x10^{-12}F/m;\ j2\pi 1x10^6 x100x10^{-12} = j0{,}00063\ S/m$

Calculate $\gamma$, α and β from the distributed constants

$$\boldsymbol{\gamma} = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}$$

$$= \sqrt{(100x10^{-3} + j1{,}696)(37x10^{-6} + j0{,}00063)} = \sqrt{-0{,}00106 + j0{,}000126}$$
$$= \sqrt{0{,}001072\angle 173{,}26°}$$
$$= \sqrt{0{,}001107\ e^{j3{,}02387}} = \left(0{,}001107\ e^{j3{,}02387}\right)^{\frac{1}{2}} = 0{,}0332\ e^{j1{,}5119} = 0{,}033\angle 86{,}67°$$
$$= \mathbf{0{,}00192 + j0{,}0331}$$

(other root 's real part is negative, meaningless here)

$$\boldsymbol{\alpha = 0{,}00192\ \frac{1}{m}\ ;\ \beta = 0{,}033\frac{rad}{m}}$$

Velocity of propagation is

$$u_p = \frac{\omega}{\beta} \approx \frac{1}{\sqrt{LC}} = \frac{2\pi 1 * 10^6}{0{,}033} = 190{,}5 * \frac{10^6 m}{s} \approx 0{,}635\ c\ (63{,}5\%\ speed\ of\ light)$$

Calculate $Z_0$ (Select root with positive real part)

$$\boldsymbol{Z_0} = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{(100x10^{-3} + j1{,}696)}{(37x10^{-6} + j0{,}00063)}} = \sqrt{2692{,}1 - j0{,}6227}$$
$$= \sqrt{2692{,}1\angle -0{,}013252°} = \sqrt{2692{,}1e^{-j0{,}000231}} = 51{,}88\ e^{-j0{,}0001156} = 51{,}89\angle -0{,}00662°$$
$$= \mathbf{51{,}88 - j0{,}0060\ \Omega}$$

Calculate $Z_{IN}$

$$\boldsymbol{Z_{IN}} = Z_0 \frac{Z_L + Z_0\left(\dfrac{\tanh(\alpha l) + j\tan(\beta l)}{1 + j\ tanh(\alpha l)\tan(\beta l)}\right)}{Z_0 + Z_L\left(\dfrac{\tanh(\alpha l) + j\tan(\beta l)}{1 + j\ tanh(\alpha l)\tan(\beta l)}\right)}$$

$$= (51{,}8 - j0{,}0060)\ \frac{50 + j30 + (51{,}8 - j0{,}0060)\left(\frac{tanh(0{,}00192x25) + j\,tan(0{,}033x25)}{1 + j\,tanh(0{,}00192x25)\tan(0{,}033x25)}\right)}{51{,}8 - j0{,}0060\ +\ (50 + j30)\left(\frac{tanh(0{,}00192x25) + j\,tan(0{,}033x25)}{1 + j\,tanh(0{,}00192x25)\tan(0{,}033x25)}\right)}$$

$$= (51{,}8 - j0{,}0060)\ \frac{50 + j30 + (51{,}8 - j0{,}0060)(0{,}104 + j1{,}080)}{51{,}8 - j0{,}0060\ +\ (50 + j30)(0{,}104 + j1{,}080)}$$

$$= (51{,}8 - j0{,}0060)\frac{55{,}39 + j85{,}94}{24{,}6 + j57{,}11} = \mathbf{84{,}0 - j14{,}1\Omega = 85{,}17\angle - 9{,}52\ \Omega}$$

Calculate $I_{IN}$

$$\boldsymbol{I_{IN}} = \frac{V_{IN}}{Z_{IN}} = \frac{100\angle 0°V}{85{,}17\angle - 9{,}52°\Omega} = \mathbf{1{,}17\angle 9{,}52°\ A}$$

Solve V+ and V-

$$V(z = 0) = V_{IN} = V^{+} + V^{-} = 100V\angle 0°$$
$$I(z = 0) = I_{IN} = \frac{V^{+}}{Z_0} - \frac{V^{-}}{Z_0} = 1{,}17\angle 9{,}52°A$$

By substitution:

$$I_{IN} = \frac{100\angle 0° - V^{-}}{51{,}89\angle - 0{,}00662°} - \frac{V^{-}}{51{,}89\angle - 0{,}00662°} = 1{,}17\angle 9{,}52°A =>$$

$$\boldsymbol{V^{-}} = \frac{100\angle 0° - 60{,}71\angle 9{,}53°}{2} = \mathbf{20{,}68\angle - 14{,}06°V} = 20{,}06 - j5{,}03\ V$$

$$\boldsymbol{V^{+}} = 100\angle 0° - 20{,}68\angle - 14{,}06° = 79{,}94 + j5{,}02 = \mathbf{80{,}09\angle + 3{,}60°V}$$

Voltage at load (z=l)

$$V(z = l) = V_L = V^{+}e^{(-\gamma l)} + V^{-}e^{(\gamma l)}$$

$$\boldsymbol{V_L} = V^{+}e^{-(\alpha l + j\beta l)} + V^{-}e^{\alpha l + j\beta l} = V^{+}e^{-\alpha l}e^{-j\beta l} + V^{-}e^{\alpha l}e^{j\beta l}$$
$$= V^{+}e^{-\alpha l}(cos\,\beta l - i\,sin\beta l) + V^{-}e^{\alpha l}(cos\,\beta l + i\,sin\beta l)$$

$$= (79{,}94 + j5{,}02)e^{-0{,}00192x25}(cos(0{,}033x25) - i\,sin(0{,}033x25))$$
$$+ (20{,}06 - j5{,}03)e^{0{,}00192x25}(cos(0{,}033x25) + i\,sin(0{,}033x25))$$
$$= (79{,}94 + j5{,}02)x0{,}953x(0{,}679 - j0{,}735) + (20{,}06 - j5{,}03)x1{,}049x(0{,}679 + j0{,}735)$$
$$= 73{,}41 - j40{,}86\ V = \mathbf{84{,}01\angle - 0{,}51°V}$$

Current at load (z=l)

$$I(z = l) = I_L = \frac{V^{+}}{Z_0}e^{(-\gamma l)} - \frac{V^{-}}{Z_0}e^{(\gamma l)}$$



P.Viljakainen
January 2021

$$\boldsymbol{I_L} = \frac{V^+e^{-\alpha l}(\cos\beta l - i\,sin\,\beta l)}{Z_0} - \frac{V^-e^{\alpha l}(\cos\beta l + i\,sin\,\beta l)}{Z_0}$$

$$= \frac{(79{,}94 + j5{,}02) * 0{,}953 * (0{,}679 - j0{,}735)}{51{,}88 - j0{,}0060} - \frac{(20{,}06 - j5{,}03) * 1{,}049 * (0{,}679 + j0{,}735)}{51{,}88 - j0{,}0060}$$

$$= \frac{37{,}08 - j64{,}63}{51{,}88 - j0{,}0060} = 0{,}715 - j1{,}246\ A = \boldsymbol{1{,}44\angle - 1{,}05°\ A}$$

Voltage at an arbitrary point (z') along the transmission line can be obtained (e.g. 10m from load)

$$V(z') = I_L[Z_L\cosh(\gamma z') + Z_0\sinh(\gamma z')]$$

Because

$$cosh(\mathrm{a} + \mathrm{jb}) = cosh(a)cos(b) + jsinh(a)sin(b)$$

and

$$sinh(a + jb) = sinh(a)cos(b) + jcosh(a)sin(b)$$

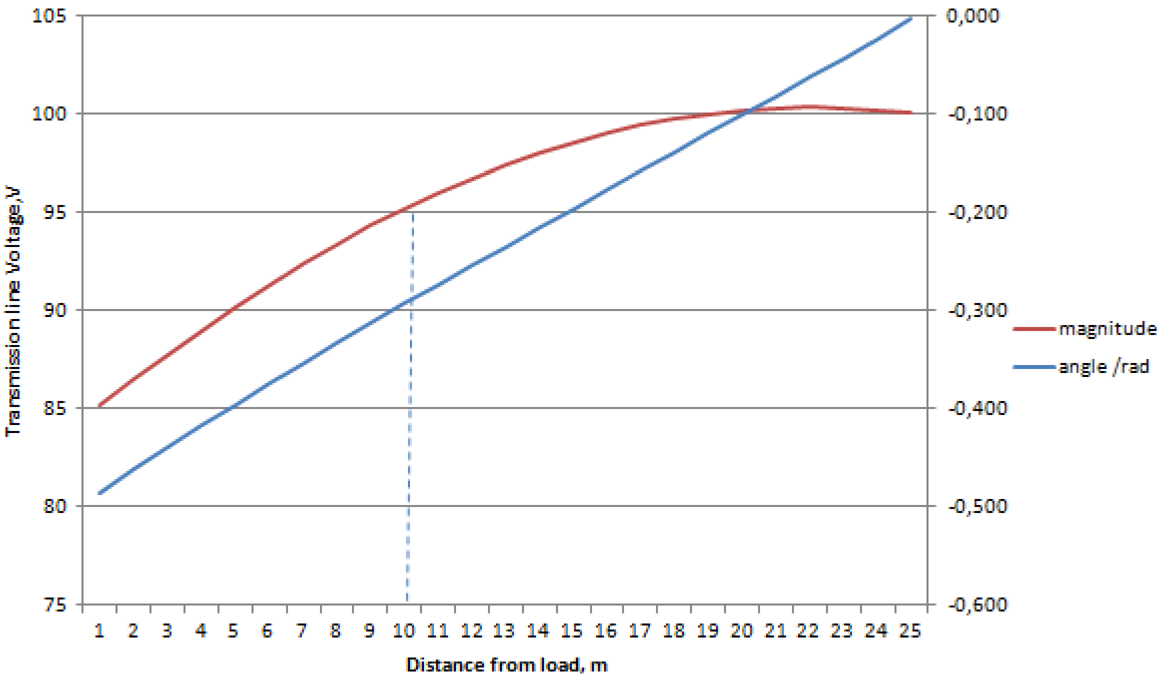
$$V(z') = I_L\big[Z_L[\cosh(\alpha z')\cos(\beta z') + isinh(\alpha z')\sin(\beta z')] + Z_0[\sinh(\alpha z')\cos(\beta z') + icosh(\alpha z')\sin(\beta z')]\big] =>$$

$\boldsymbol{V(z' = 10)} =$

$$(0{,}715 - j1{,}246)\big[(50 + j30)[\cosh(0{,}0192)\cos(0{,}33) + isinh(0{,}019)\sin(0{,}33)] + (51{,}88 - j0{,}006)[\sinh(0{,}0192)\cos(0{,}33) + icosh(0{,}0192)\sin(0{,}33)]\big]$$

$$= (0{,}715 - j1{,}246)\big[\,(50 + j30)\big((0{,}946 + j0{,}006) + (51{,}88 - j0{,}006)(0{,}018 + j0{,}324)\big)\big]$$

$$= (0{,}715 - j1{,}246)(48{,}03 + j45{,}48) = -91{,}03 - j27{,}35 = \boldsymbol{95{,}05\angle - 16{,}7°\ V}$$




$$P_{avG} = \frac{1}{2}Re[V_G I_{IN}^*]$$

$$\boldsymbol{P_{avG}} = \frac{1}{2}Re[V_G I_{IN}^*] = \frac{1}{2}Re[100\angle 0°(1{,}17\angle 9{,}52°)^*] = \frac{1}{2}Re[100\angle 0°(1{,}17\angle -9{,}52°]W = 57{,}7W$$

$$P_{avL} = \frac{1}{2}I_L^2 R_L$$

$$\boldsymbol{P_{avL}} = \frac{1}{2}I_L^2 R_L = \frac{1}{2}1{,}44^2 * 50 = \mathbf{51,8W}$$

Voltage reflection coefficient at load is

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$\boldsymbol{\Gamma_L} = \frac{(50 + j30) - (51{,}88 - j0{,}0060\,)}{(50 + j30) + (51{,}88 - j0{,}0060\,)} = 0{,}063 + j0{,}276 = \mathbf{0,283\angle 77,17°}$$

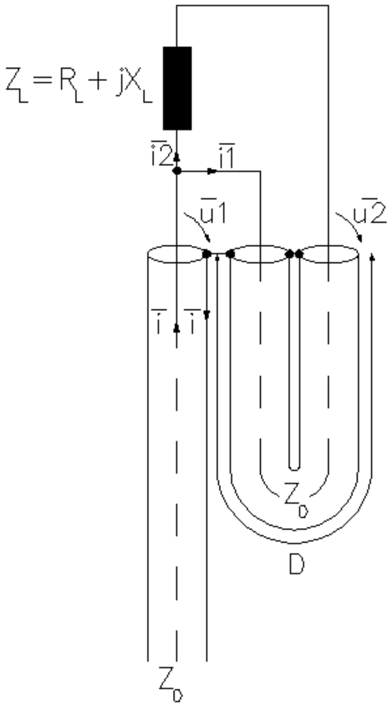
Voltage standing wave ratio is

$$S = \frac{1 + |\Gamma|}{1 - |\Gamma|}$$

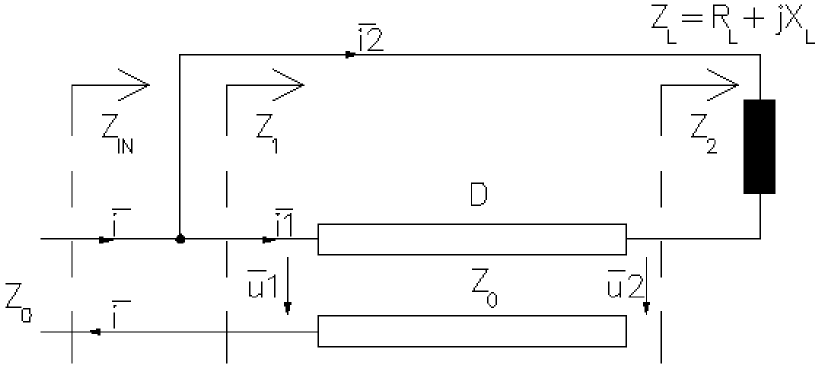
$$\boldsymbol{S} = \frac{1 + 0{,}283}{1 - 0{,}283} = \mathbf{1,79}$$

## 7. Half-wave transmission line impedance transformer

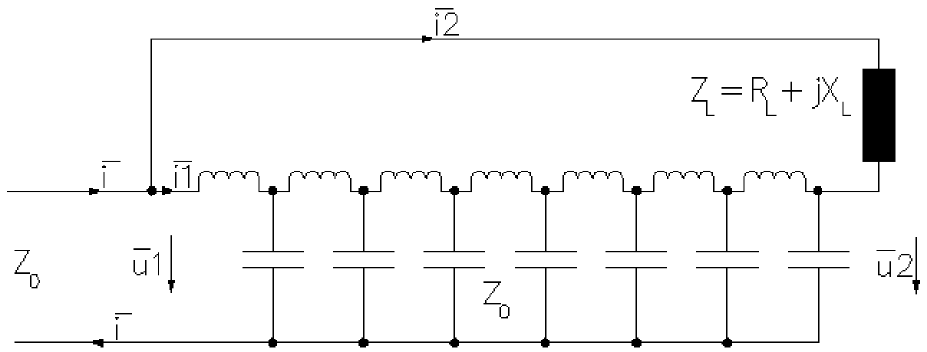
A half- wavelength transmission line can be used as an impedance transformer. In this example the transmission line is assumed to be lossless.




**Kuva 10:Half-wavelength 1:4 impedance transformer**




**Kuva 11:Half-wavelength impedance transformer, schematic**

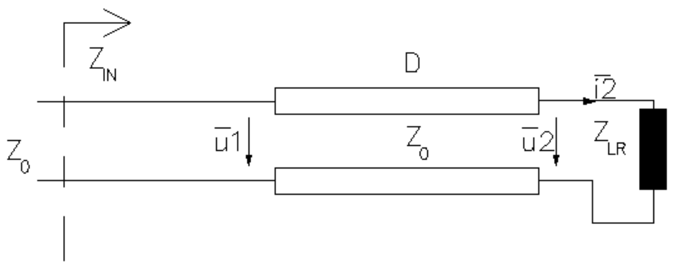



**Kuva 12:Half-wavelength impedance transformer, lumped model**

Load $Z_L$ in this case is connected across the transmission line as shown in the schematic diagram. The resulting load impedance needed in the calculation of the transmission line input impedance $Z_{IN}$ is therefore unknown, as it depends on the actual voltage $u_2$ and current $i_2$ at the end of the transmission line, which both depend on the resulting load impedance.

To be able to solve voltage and current along the transmission line an iterative method must be used.

1. Calculate $Z_{IN}$ with an seed value (guess) of "reduced" load impedance $Z_{LR}$




$$Z_{IN} = Z_0 \frac{Z_{LR} + jZ_0 tan\beta D}{Z_0 + jZ_{LR} tan\beta D}$$

2. Calculate $I_{IN}$

$$\boldsymbol{I_{IN}} = \frac{V_{IN}}{Z_{IN}}$$

3. Solve and calculate $V^+$ and $V^-$

$$V(0) = V^+ + V^-$$

$$I_{IN} = \frac{V^+}{Z_0} - \frac{V^-}{Z_0}$$



P.Viljakainen
January 2021

4. Calculate V and I at distance D (at the end of the transmission line)

$$V(D) = V^{+}(cos(-\beta D) + jsin(-\beta D)) + V^{-}(cos(\beta D) + jsin(\beta D))$$
$$I(D) = \frac{V^{+}}{Z_0}\left(cos(-\beta D) + jsin(-\beta D)\right) - \frac{V^{-}}{Z_0}\left(cos(\beta D) + jsin(\beta D)\right)$$

5. Calculate the current $i_2$ through actual load $Z_L$ across the transmission line which would be caused by $u_1$ and V(D)

$$i_2 = \frac{u_1 - V(D)}{Z_L}$$

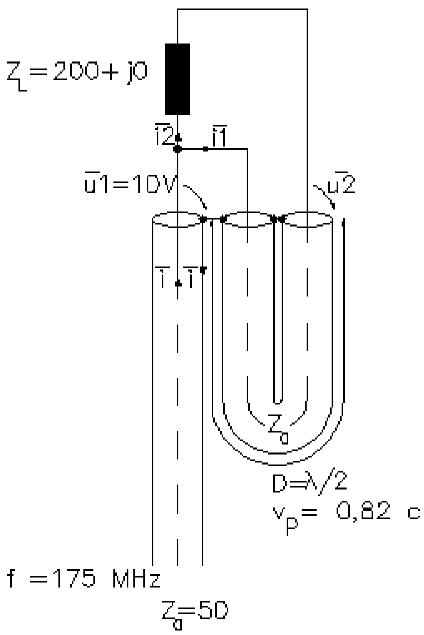
6. Calculate $-i_2$

7. Compare - i2 with I(D), both magnitude and phase shall match when "reduced" load impedance $Z_{LR}$ is correct, if no match, change the initial reduced load impedance estimate and re-calculate.

8. When –i2 and I(D) match, voltage and current can be calculated at any point of the transmission line.

## 7.1 Example

Define input impedance $Z_{in}$, voltage $u_2$, currents i, $i_1$ and $i_2$ and voltage and current along the length of the λ/2 transmission line.



β=2π/λ=2πf/$v_p$=2π175*$10^6$/(0,82*3*108)=4.4697 1/m
βD=(2π/λ)*( λ/2)=π

Calculate $Z_{IN}$ with a seed value (guess) of “reduced” load impedance $Z_{LR}$ = 70 + j0 (arbitrary)

$$Z_{IN} = Z_0 \frac{Z_{LR} + jZ_0 tan\beta D}{Z_0 + jZ_{LR} tan\beta D} = 50 \frac{70 + j0 + j50 tan\pi}{50 + j(70 + j0) tan\pi} = 70 + j0$$

$$\boldsymbol{I_{IN}} = \frac{V_{IN}}{Z_{IN}} = \frac{10\angle 0}{70\angle 0} = 0{,}14\angle 0A$$

$$V^- = \frac{u_1 - (Z_0 * I_{IN})}{2} = \frac{10\angle 0 - (50 * 0{,}14\angle 0A)}{2} = 1{,}429\angle 0\, V$$

$$V^+ = u_1 - V^- = 10\angle 0 - \; 1{,}429\angle 0\, V = 8{,}571\; \angle 0\, V$$

$$V(D) = V^+\big(cos(-\beta D) + jsin(-\beta D)\big) + V^-\big(cos(\beta D) + jsin(\beta D)\big)$$
$$= 8{,}571\; \angle 0\, V\; \big(\cos(-\pi) + \text{jsin}(-\pi)\big) + \; 1{,}429\angle 0\, V(\cos(\pi) + jsin(\pi)) = 10\angle 180\, V$$

$$I(D) = \frac{V^+}{Z_0}\big(cos(-\beta D) + jsin(-\beta D)\big) - \frac{V^-}{Z_0}\big(cos(\beta D) + jsin(\beta D)\big)$$
$$= \frac{8{,}571\; \angle 0\, V}{50}\; \big(cos(-\beta D) + jsin(-\beta D)\big) - \frac{1{,}429\angle 0\, V}{50}\big(\cos(\pi) + jsin(\pi)\big) = 0{,}143\angle 180\, A$$

$$i_2 = \frac{u_1 - V(D)}{Z_L} = \frac{10\angle 0 - \; 10\angle 180}{200 + j0} = 0{,}100\angle 0$$
$$-i_2 = -0{,}100\angle 0 = \; 0{,}100\angle 180$$

$$I(D) = 0{,}143\angle 180\, A \neq \; -i_2 = 0{,}100\angle 180 \rightarrow$$

Make a new estimate for $Z_{LR}$ = 100 +j0; do the calculation above again;

$$Z_{IN} = Z_0 \frac{Z_{LR} + jZ_0 tan\beta D}{Z_0 + jZ_{LR} tan\beta D} = 50 \frac{100 + j0 + j50 tan\pi}{50 + j(100 + j0) tan\pi} = 100 + j0$$

$$\boldsymbol{I_{IN}} = \boldsymbol{i_1} = \frac{V_{IN}}{Z_{IN}} = \frac{10\angle 0}{100\angle 0} = \boldsymbol{0{,}1\angle 0A}$$

$$V^- = \frac{u_1 - (Z_0 * I_{IN})}{2} = \frac{10\angle 0 - (50 * 0{,}1\angle 0A)}{2} = 2{,}5\angle 0\, V$$

$$V^+ = u_1 - V^- = 10\angle 0 - \; 2{,}5\angle 0\, V = 7{,}5\; \angle 0\, V$$

$$V(D) = \boldsymbol{u_2} = V^+\big(cos(-\beta D) + jsin(-\beta D)\big) + V^-\big(cos(\beta D) + jsin(\beta D)\big)$$
$$= 7{,}5\; \angle 0\, V\; \big(\cos(-\pi) + \text{jsin}(-\pi)\big) + \; 2{,}5\angle 0\, V(\cos(\pi) + jsin(\pi)) = \boldsymbol{10\angle 180\, V}$$

$$I(D) = \frac{V^+}{Z_0}\big(cos(-\beta D) + jsin(-\beta D)\big) - \frac{V^-}{Z_0}\big(cos(\beta D) + jsin(\beta D)\big)$$

$$= \frac{7{,}5\,\angle 0\,V}{50}\left(cos(-\beta D) + jsin(-\beta D)\right) - \frac{2{,}5\angle 0\,V}{50}\left(\cos(\pi) + jsin(\pi)\right) = 0{,}1\angle 180\,A$$

$$\boldsymbol{i_2} = \frac{u_1 - V(D)}{Z_L} = \frac{10\angle 0 - 10\angle 180}{200 + j0} = \mathbf{0{,}1\angle 0\,A}$$

$$-i_2 = -0{,}100\angle 0 = 0{,}1\angle 180$$

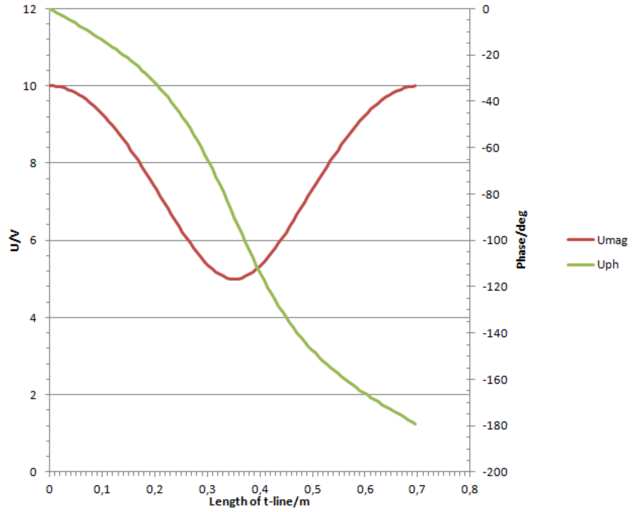
$I(D) = 0{,}1\angle 180\,A = -i_2$, so $Z_{LR}$ is correct.

$$\boldsymbol{i} = I_1 + i_2 = 0{,}1\angle 0A + 0{,}1\angle 0 = \mathbf{0{,}2\,\angle 0\,A}$$

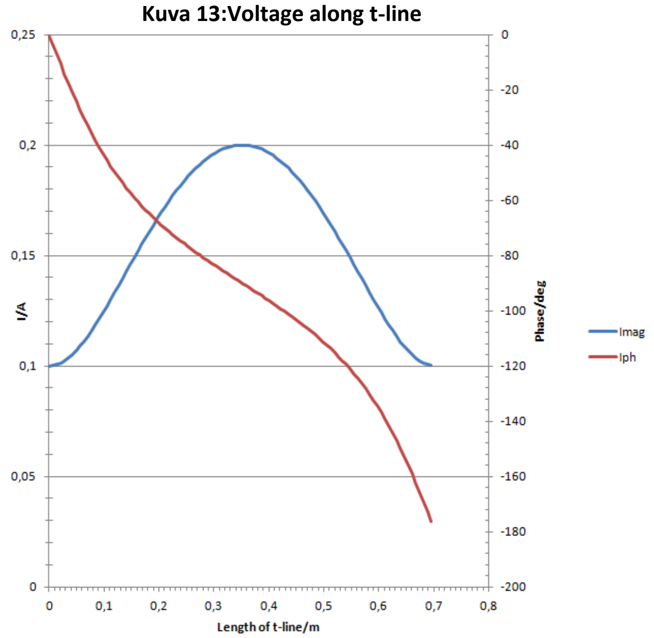
$$\boldsymbol{Z_{IN}} = \frac{u_1}{i} = \frac{10\angle 0\,V}{0{,}2\,\angle 0\,A} = \mathbf{50\Omega}$$

so Zin/ZL=1/4.

V(D) and I(D) can be calculated by varying D in the above formulas resulting in following distribution:




**Kuva 13:Voltage along t-line**




**Kuva 14:Current along t-line**